EULER CAUCHY EQUATION

The general form of second order Euler Cauchy equation is

$$x^2 y'' + axy' + by = 0 \quad \text{(Homogeneous form)}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = 0$$

$$\Rightarrow (x^2 D^2 + axD + b) y = 0 \quad \text{---------- (1)} \quad \left(D = \frac{d}{dx}\right)$$

Put $y = x^m$

$$y' = Dy = m x^{m-1}$$

$$y'' = D^2 y = m(m-1) x^{m-2}$$

Putting the values of y, y' & y'' in equation (1)

$$m(m-1) x^m + amx^m + bx^m = 0$$

$$\Rightarrow [m(m-1) + am + b] x^m = 0$$

The auxilary equation is

$$m(m-1) + am + b = 0$$

which is a second degree equation in m so it has atmost two solutions or roots.

The relation between the roots are

(I) Real unequal

(II) Real equal

(III) complex & conjugate

Case-1 (Real unequal)

Let the roots be m_1, m_2 $(m_1 \neq m_2)$

The general solution is

$$\boxed{y = C_1 x^{m_1} + C_2 x^{m_2}}$$

Scanned by CamScanner

~~Example~~

Example

Let the roots be 2, 3

The general solution is

$$y = C_1 x^2 + C_2 x^3$$

Case-II (Real equal)

Let the roots be m, m.

The general solution is

$$\boxed{y = (C_1 + C_2 \ln x) x^m}$$

Example

Let the roots be 4, 4

The general solution is

$$y = (C_1 + C_2 \ln x) x^4$$

Case-III (Complex and conjugate)

Let the roots be $\alpha \pm i\beta$.

The general solution is

$$\boxed{y = x^\alpha (C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x))}$$

Example

Let the roots be $-2 \pm 3i$

The general solution is

$$y = x^{-2} (C_1 \cos(3 \ln x) + C_2 \sin(3 \ln x))$$

Scanned by CamScanner

Page-96 E. KREYSZIG (8TH EDITION)

NO-2-TO-13

Find a general solution

2. $x^2y'' - 4xy' + 6y = 0$

$\Rightarrow x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$

$\Rightarrow (x^2D^2 - 4xD + 6)y = 0 \quad (\because D = \frac{d}{dx})$ ——(1)

Put $y = x^m$

$y' = Dy = mx^{m-1}$

$y'' = D^2y = m(m-1)x^{m-2}$

Putting these values in equation (1)

$m(m-1)x^m - 4mx^m + 6x^m = 0$

$\Rightarrow [m(m-1) - 4m + 6]x^m = 0$

The auxilary equation is

$m(m-1) - 4m + 6 = 0$

$\Rightarrow m^2 - m - 4m + 6 = 0$

$\Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - 2m - 3m + 6 = 0$

$\Rightarrow m(m-2) - 3(m-2) = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2, 3.$

The general solution is

$y = c_1x^2 + c_2x^3.$

4. $xy'' + 2y' = 0$

Multiplying x both sides,

$x^2y'' + 2xy' = 0$

Scanned by CamScanner

$\Rightarrow x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$

$\Rightarrow (x^2 D^2 + 2xD)\, y = 0 \qquad (D = \frac{d}{dx})$

$\longrightarrow (1)$

Put $y = x^m$

$y' = Dy = m x^{m-1}$

$y'' = D^2 y = m(m-1) x^{m-2}$

Putting these values in equation (1)

$m(m-1) x^m + 2m x^m = 0$

$\Rightarrow [m(m-1) + 2m] x^m = 0$

The auxiliary equation is

$m(m-1) + 2m = 0$

$\Rightarrow m^2 - m + 2m = 0$

$\Rightarrow m^2 + m = 0$

$\Rightarrow m(m+1) = 0$

$\Rightarrow m = 0, -1$

So, the general solution is

$y = c_1 x^0 + c_2 x^{-1}$

$\Rightarrow y = c_1 + c_2 x^{-1}$

14-TO-18

Solve the following initial value problems

16. $x^2 y'' + xy' + 9y = 0$, $y(1) = 2$, $y'(1) = 0$

$\Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 9y = 0$

$\Rightarrow (x^2 D^2 + xD + 9)\, y = 0 \qquad (\because D = \frac{d}{dx})$

$\longrightarrow (1)$

Put $y = x^m$

$y' = Dy = m x^{m-1}$

$y'' = D^2 y = m(m-1) x^{m-2}$

Scanned by CamScanner

Putting these values in equation (1)

$$m(m-1)x^m + mx^m + 9x^m = 0$$

$$\Rightarrow [m(m-1) + m + 9]x^m = 0$$

The auxiliary equation is

$$m(m-1) + m + 9 = 0$$

$$\Rightarrow m^2 - m + m + 9 = 0$$

$$\Rightarrow m^2 + 9 = 0$$

$$\Rightarrow m^2 = -9 = 9i^2$$

$$\Rightarrow m = \pm 3i$$

So, the general solution is

$$y = x^0 (c_1 \cos(3\ln x) + c_2 \sin(3\ln x))$$

$$\Rightarrow y = c_1 \cos(3\ln x) + c_2 \sin(3\ln x) \quad \text{---- (2)}$$

$$y' = -c_1 \frac{3}{x} \sin(3\ln x) + c_2 \frac{3}{x} \cos(3\ln x) \quad \text{---- (3)}$$

$y(1) = 2$ i.e $y = 2$ when $x = 1$ from (2)

$$2 = c_1 \cos(3\ln 1) + c_2 \sin(3\ln 1)$$

$$\Rightarrow 2 = c_1 \cos 0 + c_2 \sin 0$$

$$\Rightarrow c_1 = 2 \quad \text{---- (4)}$$

$y'(1) = 0$ i.e $y' = 0$ when $x = 1$ from (3)

$$0 = -c_1 \frac{3}{1} \sin(3\ln 1) + c_2 \frac{3}{1} \cos(3\ln 1)$$

$$\Rightarrow 0 = -c_1 3 \sin 0 + c_2 3 \cos 0$$

$$\Rightarrow 3c_2 = 0$$

$$\Rightarrow c_2 = 0 \quad \text{---- (5)}$$

Using (4) and (5) in equation (2) the solution is

$$y = 2\cos(3\ln x)$$

Scanned by CamScanner

17. $(x^2D^2 - 3xD + 4)y = 0$ ——— (1) $y(1) = 0$, $y'(1) = 3$

Put $y = x^m$

$y' = Dy = mx^{m-1}$

$y'' = D^2y = m(m-1)x^{m-2}$

Putting these values in equation (1)

$m(m-1)x^m - 3mx^m + 4x^m = 0$

$\Rightarrow [m(m-1) - 3m + 4]x^m = 0$

The auxiliary equation is

$m(m-1) - 3m + 4 = 0$

$\Rightarrow m^2 - m - 3m + 4 = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2, 2$

So, the general solution is

$y = (c_1 + c_2 \ln x)x^2$ ——— (2)

$y' = 2x(c_1 + c_2 \ln x) + (0 + \frac{c_2}{x})x^2$ ——— (3)

$y(0) = 0$ i.e $y = 0$ when $x = 1$ from (2)

$0 = (c_1 + c_2 \ln 1)1^2$

$\Rightarrow c_1 = 0$ ——— (4)

$y'(1) = 3$ i.e $y' = 3$ when $x = 1$ from (3)

$3 = 2 \times 1 (c_1 + c_2 \ln 1) + (0 + \frac{c_2}{1})1^2$

$\Rightarrow 3 = 2c_1 + c_2$ ——— (5)

Solving equation (4) and (5)

$c_1 = 0$, $c_2 = 3$

Putting the value of c_1 and c_2 to equation (2) the solution is

$y = (0 + 3 \ln x)x^2 = (3 \ln x)x^2$

Scanned by CamScanner